On the optimal control of stochastic porous media equations

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Let $\mathscr{D} \subset \mathbb{R}^n$ be a smooth bounded domain, $\beta : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and W be a Brownian motion in $L^2(\mathscr{D})$. Consider the equation

$$\begin{aligned} dx_t &= \left[\Delta\beta(x_t) + b(x_t, u_t, t)\right] dt + \sigma(x_t, t) dW_t, && (\xi, t) \in \mathscr{D} \times (0, T) \\ \beta(x_t) &= 0, && (\xi, t) \in \partial\mathscr{D} \times (0, T) \\ x_0 &\in L^2(\mathscr{D}). \end{aligned} \tag{1}$$

Equation (1) is often referred to in the literature as a stochastic porous media equation. There is by now a robust and general theory for the existence, uniqueness and regularity of solutions for equations like (1) (β doesn't even need to be a single valued function, see the monograph [BDPR16]).

We consider the problem of finding u in some space $\mathscr{U}$ of admissible stochastic processes, which minimizes a cost:

$$\bar{u} \in \arg\min_{u \in \mathscr{U}} \mathscr{J}(u), \quad \mathscr{J}(u) := \int_0^T \int_{\mathscr{D}} \ell(x_t, u_t, t) d\xi dt + \int_{\mathscr{D}} g(x_T) d\xi. \tag{2}$$

Heuristically, one could try to write a maximum principle for problem (2) by considering the adjoint equation

$$\begin{aligned} dp_t &= -\left[\beta'(x_t)\Delta p_t + b_x(x_t, u_t, t) p_t + \sigma_x(x_t, t) q_t - \ell_x(x_t, u_t, t)\right] dt + q_t dW_t, && (\xi, t) \in \mathscr{D} \times (0, T) \\ p_T &= -g_x(x_T) && \xi \in \partial\mathscr{D}. \end{aligned} \tag{3}$$

and the Hamiltonian

$$H(x, u, t, p, q) = b(x, u, t) p + \sigma(x, t) q - \ell(x, u, t).$$

Even in the deterministic case (where we omit all terms involving W, σ, q), this is far from direct, as the coefficient $\beta'(x_t)$ depends on the solution of the state equation. One approach (see for instance [Mar06] and [MT06]), is to approximate β by a smoother function β_ε and pass to the limit in the optimality conditions for the "smoothed" problem.

In the stochastic case, we have the additional complication that backward-time stochastic evolution equations are fundamentally different from their forward-time counterparts and there is no comprehensive theory to treat them.

In this talk, we will discuss some of the difficulties that arise in trying to provide optimality conditions for problem (2).

[BDPR16] Viorel Barbu, Giuseppe Da Prato, and Michael Röckner. *Stochastic porous media equations*, volume 2163. Springer, 2016.

[Mar06] Gabriela Marinoschi. *Functional approach to nonlinear models of water flow in soils*, volume 21. Springer, 2006.

[MT06] Zhou Meike and Dan Tiba. Optimal control for a stefan problem. In *Analysis and Optimization of Systems: Proceedings of the Fifth International Conference on Analysis and Optimization of Systems Versailles, December 14–17, 1982*, pages 776–787. Springer, 2006.